Out-of-equilibrium dynamics in infinite one-dimensional self-gravitating systems

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Collaborations and Papers

One dimensional gravity in infinite point distributions

A dynamical classification of the range of pair interaction

Non-linear gravitational clustering of cold matter in an expanding universe: indications from 1D toy models
Outline

1. Motivation
2. 1D Toy models
3. Numerical Methods
4. Results
5. Conclusion and Perspectives
Motivation: Understanding of Structure Formation in the Universe

Observations:

“WMAP” (10^5 years): density fluctuations $\sim 10^{-5}$ ...

... and “SDSS” (10^{10} years): density fluctuations $\gg 1$. 
The standard model of cosmology:

Observed large scale structures are generated (essentially) by action of gravity alone starting from a close-to-uniform state. Further most (> 80%) matter is assumed to be

Initially “COLD”: non-relativistic

“DARK”: one only considers gravitational interaction

CDM makes structures more efficiently than ordinary matter

Does the standard theory work (i.e. reproduce observations)?

Dark matter is not visible!

“bias” between galaxies and dark matter
Motivation: Power-law scaling and scale-invariance in galaxy clustering

Observations:

Power law behaviours characterize galaxy correlations

Frequently asked questions:

- is such power-law clustering in galaxies indicative of scale-invariant phenomena?
- if yes, can gravitational dynamics alone give rise to it?
Conclusions for standard model based on:

- Limited analytical understanding: “linear theory”
- Numerical results from “N-body” simulations
- Phenomenological models (e.g. “halo models”)

Questions:

- Are the numerical results for distribution of DM correct?
- Could scale invariance arise from this dynamics?
Work in the limit of Newtonian gravity:

Classical self-gravitating particles in an infinite universe
IC: small perturbations around uniform distribution of points

We discuss here:

Dynamics of 1D version of 3D systems and analogy
Why 1D models?

**Approach:** Statistical Physics

Toy models

Fundamental aspects

Out-of-equilibrium dynamics of LRI

**Relevance:**

Exact integration of equation of motion

Accuracy limited by machine precision

Spatial resolution $\sim N^{1/d}$
Cosmological structure formation:

1D TOY MODELS
Definition & Motivation
Newtonian Gravity in 1D finite system

1D 2-body potential: $\Phi(x) = g \ |x|$

$$F(x) = -\frac{\partial \Phi}{\partial x} = -g \ \text{sgn}(x)$$

→ attractive pair force independent of separation

In a system of a **FINITE** number of particles ($i = 1 \ldots N$)

$$F(x) = -g \sum_i \text{sgn}(x - x_i)$$

$$F(x) \propto N_R(x) - N_L(x)$$
**Infinite** uniform system in cosmology (static/expanding)

\[ N \to \infty, \ L \to \infty, \ \frac{N}{L} = n_0 = \text{constant} \]

- The force is **ill defined**

\[ F(x) \propto \lim_{L \to \infty} \left[ N_R(x, L) - N_L(x, L) \right] \]

- **Remove (divergent) contribution of mean density**

\[ F(x) = -g \lim_{\mu \to 0} \sum_i \text{sgn}(x - x_i) e^{-\mu |x - x_i|} \]
Particles displaced from infinite perfect lattice configuration

The (regularized) force in 1D on a particle

\[ F(u) = 2gn_0u \quad \text{if particles do not cross} \]

Smooth crossing = hard elastic collision + exchange of label
Displacements of $n^{th}$ particle: $u_n = x_n - n\ell$

Microscopic number density:

$$n(x) = \sum_{n=-\infty}^{+\infty} \delta_D(x - n\ell - u_n)$$

Screened Force:

$$F_{\mu}(x_0) = g \sum_{n\neq 0} \text{sgn}(x_n - x_0) e^{-\mu|x_n-x_0|}$$

Approach introduced by Chandrasekhar: PDF of the Force

$$P(F; u_0) = \lim_{\mu \to 0} P_\mu(F_\mu; u_0) = \delta_D\left(F - 2g(u_0 - \langle U\rangle)\right)$$
Cosmological simulations: equations of motion

**static universe in 1D:**

\[ \ddot{u}_i(t) = 2gn_0 u_i(t) \]

**expanding universe in 1D:**

\[ \ddot{u}_i + 2H \dot{u}_i = \frac{2gn_0}{a^3} u_i \]

→ Derived by analogy with 3D
→ \( a(t) \) is “scale factor” for cosmological model
→ \( H(t) = \frac{1}{a} \frac{da}{dt} \) represents the “Hubble expansion”

**Initial condition:**

→ Particles displaced from an infinite regular lattice
In reference EdS cosmology: $a(t) \propto t^{2/3}$ & $H^2 = \frac{8\pi G \rho_0}{3a^3}$

The change of variable $\tau = \sqrt{2/3} \ln(t)$ gives

$$\frac{d^2 u_i}{d\tau^2} + \frac{1}{\sqrt{6}} \frac{du_i}{d\tau} = u_i$$

→ expansion $\sim$ fluid damping

→ General expression

$$\frac{d^2 u_i}{d\tau^2} + \Gamma \frac{du_i}{d\tau} = u_i$$

$\Gamma = 1/\sqrt{6} \equiv$ "Quintic" Model

$\Gamma = 1/\sqrt{2} \equiv$ "RF" Model (Cubic Model)
Cosmological simulations in $3D$ and $1D$: METHODS
**Implementation**: infinite periodic system ($N$ particles per cell)

**Initial conditions**: perturbed lattices

**Displacements of particles**: realization of stochastic process, chosen to give desired power spectrum of density fluctuations $P(k)$

**Study power law IC**: 

\[ P(k) \propto k^n, \quad n=0, 2, 4 \quad \text{at small } k \]

**Focus on a particularly simple case**: “shuffled lattice”

\[ P(k) \propto k^2 \quad \text{at small } k \]
Between collisions, solution of equations of motion trivial
→ Solve algebraic equations to determine next pair collisions
→ Update positions and velocities to this time
→ Exchange velocities (particles “colliding”)
→ Iterate

Short-cut: event-driven “heap algorithm”
(log N operations to determine next pair crossing)
1 Dimension:
- No need to discretize equations of motion
- Only limitation on accuracy is machine precision
- No intrinsic limit on spatial resolution

3 Dimensions:
- Regularization with smoothing
- Numerical cost for calculation of the force
- Discretization of equation of motion
Cosmological gravitational clustering in 1D (and 3D):

RESULTS
Hierarchical clustering in 3D from Shuffled lattice IC

Results: Hierarchical clustering in a 1-d universe
Linear amplification of fluctuations

Evolution of density fluctuations of low amplitude:

→ **linearized hydrodynamic equations**

\[
\frac{\rho(x,t) - \rho_0}{\rho_0} = \delta(x,t) = A(t) \, \delta(x, t = 0)
\]

→ \( A(t) \) depends on static/expanding

**In reciprocal space:**

\[
\hat{\delta}_\rho(k, t) = A(t) \, \hat{\delta}_\rho(k, t = 0)
\]

→ Wavenumber-independent amplification of fluctuations.

**For power spectrum (structure factor):**

\[
P(k) \propto |\hat{\delta}_\rho(k, t)|^2 \quad \text{and} \quad P(k, t) = A^2(t) \, P(k, t = 0)
\]
Results: Evolution of PS (n=2, static model 1D)

\[ n_0 = N = 1/\ell_{lat} \quad \& \quad k_N = \pi/\ell_{lat} \]
$n_0 = N = 1/\ell_{\text{lat}} \quad \& \quad k_N = \pi/\ell_{\text{lat}}$
Results: Evolution of PS (n=2, static model 3D)

Results: Evolution of PS (n=2, EdS model 3D)

Results: Evolution of Correlation function (1D)

\[ \xi(x/L, t) \]

\( x/L \)

\( t = 0 \)
\( t = 4 \)
\( t = 6 \)
\( t = 8 \)
\( t = 10 \)
\( t = 12 \)
Results: Evolution of Correlation function (3D)

Important (numerical) result in 3D is “self-similarity”

For power law IC, one observes asymptotically

$$\xi(x, t) \approx \xi\left(\frac{x}{R_s(t)}\right)$$

→ Evolution $\sim$ Spatial rescaling
Results: Correlation function & Self-similarity (1D)
Important (numerical) result in 3D is “self-similarity”

For power law IC, one observes asymptotically

\[ \xi(x, t) \approx \xi_0 \left( \frac{x}{R_s(t)} \right) \]

\[ \rightarrow \text{Evolution} \sim \text{Spatial rescaling} \]

\[ \rightarrow R_s(t) = A(t)^{2/(d+n)} \text{ derived from linear theory:} \]
Characterization of non-linear clustering in 1D
Determination of correlation exponent (1D)

\[ \gamma = \frac{\ln \xi_{\text{min}} - \ln \xi_{\text{max}}}{\ln x_{\text{min}} - \ln x_{\text{max}}} \]

\[ \xi(x) \sim x^{-\gamma} \]
Correlation exponent determined as:

\[ \gamma = \frac{\ln \xi_{\text{min}} - \ln \xi_{\text{max}}}{\ln x_{\text{min}} - \ln x_{\text{max}}} \]

→ Behaviour of upper cut-off:

\[ x_{\text{max}} \propto R_s(t) \quad \& \quad \xi_{\text{min}} \sim 1 \]

→ Behaviour of lower cut-off \( x_{\text{min}} \) and \( \xi_{\text{max}} \) ?
Determination of correlation exponent (1D)
Determination of the lower cut-off (1D)

Measure of the lower cut-off (Numerical investigation)

\[ x_{\text{min}} \propto \xi_{\text{max}}^{-1} \propto \exp\left(-\frac{2}{3} \Gamma t\right) \]

Stable clustering in 1D (Finite Overdensity)

\[
\frac{d^2 u_i}{dt^2} + \Gamma \frac{du_i}{dt} = u_i
\]

\[
\frac{d^2 (x_i - x_{CM})}{dt^2} + \Gamma \frac{d(x_i - x_{CM})}{dt} = \left(\frac{N_i^> (t) - N_i^< (t)}{2n_0}\right) + (x_i - x_{CM})
\]
Finite overdensity in an infinite universe

Virialised overdensity

Adiabatic approximation

\[ \frac{dE}{dt} = -\Gamma \left( \frac{dx}{dt} \right)^2 = -2\Gamma K \]

\[ \langle E \rangle = \langle K \rangle + \langle U \rangle \]

\[ \langle L_s \rangle \propto \exp\left(-\frac{2}{3} \Gamma t\right) \propto x_{min} \]

In agreement with Numerical simulation
Determination of correlation dimension (1D)

Right understanding & Generalization of stable clustering

\[ \gamma = \frac{2\Gamma(n + 1)}{\Gamma(2n - 1) + 3\sqrt{\Gamma^2 + 4}} \]

which agrees very well with observed values

<table>
<thead>
<tr>
<th>initial PS</th>
<th>Quintic</th>
<th>RF</th>
<th>Quintic (simulation)</th>
<th>RF (simulation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 0</td>
<td>$\gamma = 1/7$</td>
<td>$\gamma = 1/4$</td>
<td>$\gamma = 0.14 \pm 0.02$</td>
<td>$\gamma = 0.25 \pm 0.02$</td>
</tr>
<tr>
<td>n = 2</td>
<td>$\gamma = 1/3$</td>
<td>$\gamma = 1/2$</td>
<td>$\gamma = 0.35 \pm 0.02$</td>
<td>$\gamma = 0.50 \pm 0.02$</td>
</tr>
<tr>
<td>n = 4</td>
<td>$\gamma = 5/11$</td>
<td>$\gamma = 5/8$</td>
<td>$\gamma = 0.43 \pm 0.01$</td>
<td>$\gamma = 0.62 \pm 0.01$</td>
</tr>
</tbody>
</table>
Non-linear clustering and Scale invariance in 1D
Non-linear clustering in 1D: Visual inspection

Whole system ($N = 10^5$ particles)
Non-linear clustering in 1D: Visual inspection

1/10 th of system
Non-linear clustering in 1D: Visual inspection

$1/10^2$ of system
Non-linear clustering in 1D: Visual inspection

$1/10^3$ of system
1/10^4 of system
Non-linear clustering in 1D: Visual inspection

$\frac{1}{10^5}$ of system
Results: scale-invariance in 1D?

→ Study (multi-)fractal exponents using **box-counting** technique

→ Confirms strong evidence for **scale-invariance**
Results: scale-invariance in 1D?

**Static case:**
- homogeneous fractal
- exponent independent of initial condition

**Expanding case(s):**
- weakly multifractal
- exponents depend also on initial condition (i.e. on exponent in initial power spectrum)
Are there “halos” in 1D?

Are there Halos in 1D?
Identification of Halos: Friend of Friend algorithm

\[ d < l_{\text{fof}} \]
\[ d > l_{\text{fof}} \]
\[ d > l_{\text{fof}} \]
\[ d > l_{\text{fof}} \]

Qualitative and Quantitative analysis of the distribution of:

- the size of the FoF-Halos
- the density of the FoF-Halos
- the nearest Halo distance
- the virial ratio
Virial ratio of Halos in 1D

Statistical analysis with a Kolmogorov-Smirnov test
Conclusion:

- reproducible signal in the appropriate range
- statistically virialised structures
- result in line with “clustering hierarchy” envisaged by Peebles

Transposition to 3D:

- CDM Halos not well modeled as smooth objects
- Universality profile is an artefact of poor numerical resolution

M. Joyce and F. Sicard
Non-linear gravitational clustering of cold matter in an expanding universe: indication from 1D toy models
CONCLUSION & PERSPECTIVES
• **Qualitative features of 3D dynamics**
  → Hierarchical clustering, Self-Similarity, Power-law behavior

• **Large range of scales in 1D**
  → Scale-invariance, Exponents, Multifractal

• **Analysis in term of “Halos”**
  → Virialised Fractal Hierarchy
  → Very different to 3D description
Non-linear clustering in 1D qualitatively different (?)
Standard interpretation of 3D numerical results inappropriate (?)

1D results suggest paths for extracting correct 3D behavior

Smoothing in 1D

*A dynamical classification of the range of pair interaction*

Classification of pair-interaction based on convergence of the force
Study of different *long-range interaction* numerically implemented with *GPU* programming